

# Chaotic Jets

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## Abstract

The problem of characterizing the origin of the non-Gaussian properties of transport resulting from Hamiltonian dynamics is addressed. For this purpose the notion of chaotic jet is revisited and leads to the definition of a diagnostic able to capture some singular properties of the dynamics. This diagnostic is applied successfully to the problem of advection of passive tracers in a flow generated by point vortices. We present and discuss this diagnostic as a result of which clues on the origin of anomalous transport in these systems emerge.

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## 1 Introduction

The characterization of the kinetics emerging from Hamiltonian dynamics has been a long going problem which dates back to Boltzmann. As the literature evolves, more and more Hamiltonian systems are showing anomalous properties, in the sense that their kinetics does not follow a simple Gaussian process but rather gives rise to what one now calls “strange kinetics” [1,2]. In systems which belong to the of  $3/2$  degree of freedom Hamiltonians, the origin of these anomalous properties can be relatively well understood when a portrait of the

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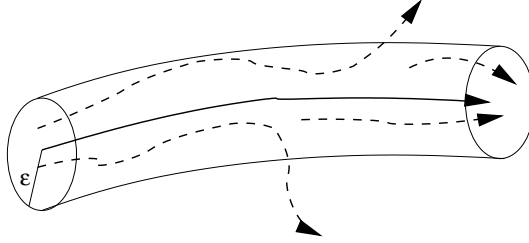


Fig. 1. Tracking of  $\epsilon$  coarse-grained regular jet.

phase space using Poincaré maps is drawn. The system is not ergodic, and a well-defined stochastic sea filled with various islands of regular motion is observed. The anomalous properties and their multi-fractal nature are then linked to the existence of islands within the stochastic sea and the phenomenon of stickiness observed around them.

One of such systems corresponds to the problem of the advection of passive particles in flows generated by three vortices. [3,4,5]. Special islands also known as “vortex cores” are surrounding each of the three vortices. Transport in these systems is anomalous, and the exponent characterizing the second moment exhibit a universal value close to  $3/2$ . Special interest in these point vortex systems follows from different observations and models that have exhibited anomalous transport properties[6,7,8,9,10,11]. The applications to geophysical flows where advected quantities vary from the ozone in the stratosphere to various pollutants coupled with the rise of the environmental concerns, make the understanding of these anomalous properties even more crucial. Systems governed by three point vortices give rise to the phenomenon know as chaotic advection, the quasi-periodic flow allows the existence of Lagrangian chaos which enhances considerably the mixing properties. The non-uniformity of the phase space and the presence of islands have a considerable impact on the transport properties. The phenomenon of stickiness on the boundaries of the islands generates strong “memory effects” as a result of which transport becomes anomalous. However, when the flows is itself chaotic and one cannot easily draw a phase portrait, it is crucial to define a proper diagnostic which will be able to capture some singular properties of the dynamics that would give clues on the origin of the anomalous transport.

There have been already different attempt to create a tool able to track this phenomenon [12,13,14]. Most of them originates from the definition of the Lyapunov exponent

$$\sigma_L = \lim_{r(0) \rightarrow 0} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \frac{r(\tau)}{r_0}, \quad (1)$$

where  $r_0$  is the initial separation between two nearby trajectories and  $r(\tau)$  is the separation at time  $\tau$ . When the dynamics is ergodic this exponent gives

a global (non-local) signature of the degree of chaoticity of the system. For instance, Finite-Time Lyapunov exponents (FTLE) are used. These exponents are measured from trajectories whose initial conditions are covering the plane, resulting in a scalar field distributed within the space of initial conditions. Regions of vanishing FTLE can then be identified as sources of “long memory” effects [12]. The difficulties with these types of approach resides in the introduction of arbitrary choice free parameters when computing FTLE, namely the initial separation between two different trajectories  $r_0$  and the time interval  $\tau$  within which they are computed: more or less arbitrary specific scales are set both in time and space. It is likely that  $r(t)$  is not always smooth growing function of time on the scale of an arbitrary time  $\tau$  and jumps between different spatial scales, each with a potential physical meaning. We can anticipate that this may be especially the case when different regions of small (if not zero) Lyapunov exponents are present in the system. Moreover since the field of FTLE is computed within the initial conditions space, this method is rather numerically expensive if one wishes to follow the regions of vanishing FTLE through time.

In the following we discuss a diagnostic which is greatly inspired from a natural phenomena typically observed in geophysical flows, namely the presence of jets. Indeed one can picture a jet as an ensemble of fluid particles traveling “coherently” together and exhibiting on a given scale little dispersion. This phenomenon does not discard the possibility of strong chaotic motion within the jet, but restrict it within a specific scale. This approach is in fact under certain aspect very similar to measuring regions of vanishing FTLE but has the advantage of clearing out some of its shortcomings. In a non ideal situation, typically when dealing with numerical or experimental data, we only have a finite spatial resolution and have only access to a finite portion of a trajectory (finite time). In some sense we are facing a “coarse grained” phase space, and each point is actually a ball from which infinitely many real trajectories can depart. Hence, two nearby real trajectories may diverge exponentially for a while but then get closer again without actually leaving our resolution scale, a process which may happen over and over; in this “coarse grained” perspective those two real trajectories are identical. It may then possible that within the phase space exist nearby trajectories which remain within our scale of interest for a relatively large time, giving rise to what we call a *chaotic jet, or simply jet* [15,16,17,?]. In this situation we are typically interested on the chaotic properties of the system from the resolution scale and up and dismay any chaotic motion which may occur within the jet.

We shall now discuss the strategy implemented to detect jets. Let us consider a trajectory  $\mathbf{r}(t)$  within the phase space. We associate to this trajectory its corresponding “coarse grained” equivalent in the following way: for each time  $t$ , we consider a ball  $B(\mathbf{r}(t), \epsilon)$  of radius  $\epsilon$  whose center is the position  $\mathbf{r}(t)$ . The  $\epsilon$ -coarse grained trajectory is then formed by the reunion of the balls for

all time  $\cup B(\mathbf{r}(t), \epsilon)$  and defined by our minimal scale of interest  $\epsilon$ . Once this  $\epsilon$ -coarse grained trajectory is defined, we look for real trajectories within the ball at a given time. We then measure two quantities: a time  $\tau$  and length  $s$ , corresponding to actually how long the trajectory remains and how much it travels before its first escape from the coarse grained trajectory (see Fig. 1).

This approach has already been used with success when studying numerically the advection of passive tracers in flows governed by point vortices [16]. In this setting, the velocity field generated by the chaotic motion of four point vortices was considered and the jet data was collected as follows: given an initial condition of a tracer, test particles are placed in its neighborhood (typically two), at a distance  $\delta = 10^{-6} \ll \epsilon = 0.03$ .  $\epsilon$  has been chosen to be typically much smaller than the characteristic small scale of the system, namely the radius of the core surrounding the vortices, which was estimated around 0.25. In this setting each time a test particles reaches a distance  $\epsilon = 0.03$  (the width of the coarse grained trajectory), the time interval  $\tau$  and the distance traveled  $s$  are recorded, the test particle is then discarded, a new one takes its place at a distance  $\delta$  and so on. The main difficulty in using this diagnostic follows from the fact that data acquisition is not sampled linearly in time nor space. Thus a careful choice for the values of  $\epsilon$  and  $\delta$  is necessary in order to avoid accumulating too much data while still capturing the rare events. Once the parameters are defined, we first look at the trapping time distribution. A power law decay with typical exponent  $\gamma = 2.82$  . is observed in in Fig. 2 for

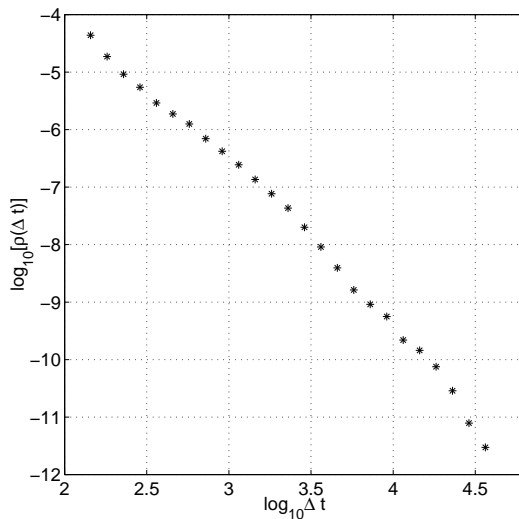


Fig. 2. Tail of the distribution of trapping times  $\Delta t$ . A power-law decay, with some oscillations is observed. Typical exponent is  $\rho(t) \sim t^{-\gamma}$  with  $\gamma \approx 2.823$ .

the system driven by four vortices. The initial condition of the vortex system is identical as the one used in [18]. The data corresponds to 4 different real trajectories. The time of the simulation is  $5.10^6$ , the time step is 0.05, and the evolution of the vortices and passive tracers was computed using a fifth order Gauss-Legendre symplectic scheme[19]. This power law shows that long

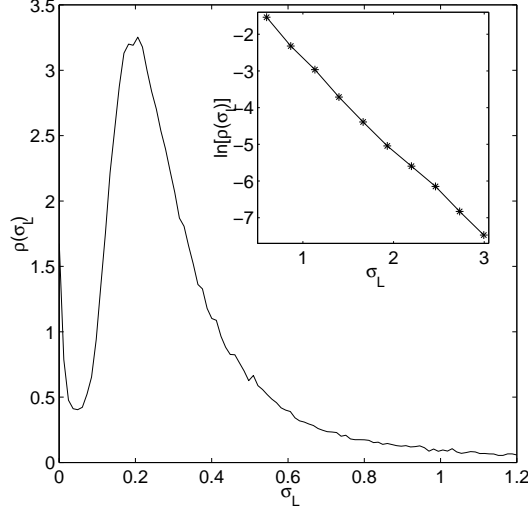


Fig. 3. Distribution of  $\sigma_L$  (see Eq. (2)). We notice an exponential decay for high exponents.  $\rho(\sigma_L) \sim \exp(-\sigma_L/\sigma_{L_0})$  with  $\sigma_{L_0} \approx 0.4$ . We can notice a minimum around  $\sigma_L \approx 0.05$ . The observed accumulation near 0 results from the existence of long lived jets.

lived jets exists and are responsible for the anomalous transport properties observed in this system (see [16] for details).

In order to identify dynamically that a reference tracer is located within a jet it is convenient to go back the definition of the Lyapunov exponent (1), which inspires two different types of FTLE:

$$\sigma_L = \frac{1}{\Delta t} \ln \frac{\epsilon}{\delta}, \quad \sigma_D = \frac{1}{\Delta s} \ln \frac{\epsilon}{\delta}, \quad (2)$$

where contrary to the typical FTLE's the value of the logarithm is fixed and  $\Delta t$  or  $\Delta s$  are the variables. These exponent are very similar to the notion of Finite Size Lyapunov Exponent (FSLE) introduced in [20], however no averages are performed and the whole distribution is used. The distribution of the measured  $\sigma_L$  are illustrated in Fig. 3. One can notice in this plot two different types of behavior: For large exponents the distribution decays exponentially with a characteristic exponent  $\sigma_{L_0} \approx 0.4$ . Since the speed of tracers is bounded, it will always take a finite time to escape from the ball, thus an expected maximum value for  $\sigma_L$ . Regarding the exponential decay behavior before reaching this maximum value, since we are measuring escape times from a given (moving) region of the phase space, this exponential behavior can be expected from a chaotic system with good mixing properties. On the other hand the existence for the small FTLE's of a local minimum in the probability density function for a non zero value of  $\sigma_L$  is more interesting when dealing with anomalous transport behavior. Indeed, this minimum characterizes the crossover from the erratic chaotic motion of the reference tracer to its motion within a regular jet.

Indeed if the tracer is within jet, the test particles are nevertheless expected to escape from the tracers vicinity but with trapping times exhibiting a power-law decay, therefore if the passive tracer is evolving within a jet for a long time, we should expect an accumulations of events corresponding to test particles leaving the jet.

The shape of the distribution of  $\sigma_D$  is qualitatively identical to the one obtained for  $\sigma_L$  in Fig. 3, with an exponential decay and a local minimum  $\sigma_{D*} \sim 0.03$  near zero. In fact the measure of  $\sigma_L$  is biased towards jets within which the average speed is slow, while by using  $\sigma_D$  these dynamical differences are erased and only the actual topology of the vicinity of a trajectory matters. Anomalous transport properties are most often characterized by measuring the time evolution of the distribution of some physical coordinate or position, which has the dimension of a length. In this light it should be clear that the actual length of jets are going to play an active role the shape of the distribution of displacements, while the role by the time spent within a jets is more subtle as it is also dependent on the speed within the jet. Hence  $\sigma_{D*}$  is the adequate parameter to control whether or not the reference tracer is evolving within a coherent jet, while its averaged speed  $\sigma_L/\sigma_D$  allows to differentiate between different types of jets (see [16] for details).

In order to dynamically look for coherent jets, a tracer's evolution with its test tracers is studied, once the threshold given by  $\sigma_{D*}$  is reached, the measured  $\sigma_D$  will be such that  $\sigma_D < \sigma_{D*}$ , hence the reference tracer is evolving in a coherent a jet. Note that the escaping of the test particles does not mean that the tracer is not still trapped within the regular jet. It is also possible to dynamically chose specifically among possible different types of jets using the averaged speed  $\sigma_L/\sigma_D$  within the jet (see [16]).

Having also access to the position of the test particles it becomes possible to gather some information about the inner structure of the jet while. The structure is illustrated by plotting the relative position of a test particles in the frame moving with the reference tracer in Fig. 4. We can effectively see a structure within the jet, and a hierarchy of circular (tubular) jets within jets emerge. This structure is somewhat robust as the same type of structure is observed when considering a system with 16 vortices [16]. Note that test particles are coming very close to the reference trajectory and that this is not an artifact of having initially placed the ghost in the vicinity of the tracer. Indeed the jet is detected only when  $\sigma_{D*}$  is reached, and for this particular plot the test particle is not in the close vicinity of the reference tracer at this moment; the influence of the parameter  $\delta$  becomes in this regards less relevant. This hierarchical structure is reminiscent of the discrete renormalization group, and we can speculate that log-periodic oscillation described in [21] may be observed. Moreover, the hierachy allows to clear out the eventual influence of the parameter  $\epsilon$ .

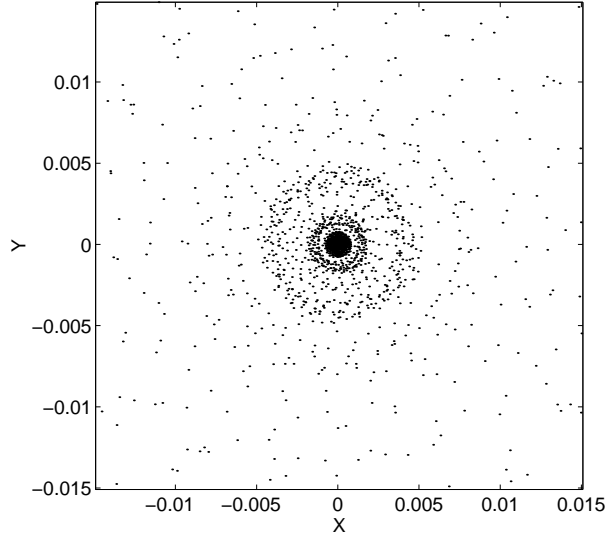


Fig. 4. Structure of a jet in the flow generated by four vortices. The distribution is not uniform and exhibits a nested set of jets with increasing radii.

In this paper we have reviewed in details the notion of chaotic jet. This notion allows to define a proper diagnostic which was able to capture the singular dynamics responsible for anomalous transport in systems of point vortices. The possibility to actually visualize the structure of the jet allows to discard the qualitative influence of the parameters  $\epsilon$  and  $\delta$ , making jets an actual robust feature of the system dynamics related to some kind of “hidden order”. In these systems, the jets were located on the boundaries of coherent structures. We therefore speculate that this behavior is generic and that the detection of jets should lead to the localization of the so-called coherent structure responsible for anomalous transport in more complex systems. The detection of jets in systems exhibiting typical properties of two-dimensional turbulence is currently under investigation. Finally we emphasize that the diagnostic we set up in order to detect coherent jets can be understood as a particular case of measurements of space-time complexity[17].

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